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# ON THE RELATION BETWEEN CAUCHY'S NUMBERS AND BESSEL'S FUNCTIONS.

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It is sometimes of advantage in Celestial Mechanics to transform formulas expressed by means of Cauchy's numbers into a form where Bessel's functions are the instrument of calculations and *vice versa*. It is therefore useful to know what relations there exist between Cauchy's numbers and Bessel's functions.

In a note on Cauchy's numbers by the author\* it has been shown that

$$N_{-p, j, q} = \left[ \frac{j}{n} \right] - \left[ \frac{j}{n-1} \right] \left[ \frac{q}{1} \right] + \left[ \frac{j}{n-2} \right] \left[ \frac{q}{2} \right] - \dots + (-1)^n \left[ \frac{q}{n} \right] \quad (1)$$

where  $n = \frac{1}{2}(-p + j + q)$ , and the symbol  $\left[ \frac{a}{b} \right]$  stands for

$$\frac{a(a-1)\dots(a-b+1)}{b!}.$$

This formula gives for  $j = 0$

$$N_{-p, 0, q} = (-1)^n \left[ \frac{q}{n} \right] = (-1)^n \frac{q!}{n! (q-n)!} = (-1)^n \frac{q!}{n! (n+p)!}. \quad (2)$$

On the other hand we have for Bessel's function  $J_i(x)$  the expression

$$J_i(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\left[ \frac{x}{2} \right]^{i+2m}}{m! (i+m)!}. \quad (3)$$

Therefore we can write

$$J_i(x) = \sum_{m=0}^{\infty} \frac{N_{-i, 0, i+2m}}{(i+2m)!} \left[ \frac{x}{2} \right]^{i+2m} \quad (4)$$

and, differentiating with regard to  $x$ ,

$$\frac{dJ_i(x)}{dx} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{N_{-i, 0, i+2m}}{(i+2m-1)!} \left[ \frac{x}{2} \right]^{i+2m-1}. \quad (5)$$

As an application of these formulas some properties of Bessel's functions

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\* Annals of Mathematics, Vol. 10, p. 1.

can be derived from the properties of Cauchy's numbers. Thus, for example, by (4)

$$J_{-i}(x) = \sum_{m=0}^{m=\infty} \frac{N_{i,0,-i+2m}}{(-i+2m)!} \left[ \frac{x}{2} \right]^{-i+2m} \quad (6)$$

and, on account of

$$N_{-p,j,q} = (-1)^p N_{p,j,q}$$

formula (6) becomes

$$J_{-i}(x) = (-1)^i \sum_{m=0}^{m=\infty} \frac{N_{-i,0,-i+2m}}{(-i+2m)!} \left[ \frac{x}{2} \right]^{-i+2m},$$

or, since  $N_{-p,j,q} = 0$  for  $-p+j+q < 0$ , and therefore  $N_{-i,0,-i+2m} = 0$  for  $m < i$ ,

$$J_{-i}(x) = (-1)^i \sum_{m=i}^{m=\infty} \frac{N_{-i,0,-i+2m}}{(-i+2m)!} \left[ \frac{x}{2} \right]^{-i+2m} = (-1)^i \sum_{m=0}^{m=\infty} \frac{N_{-i,0,2m+i}}{(2m+i)!} \left[ \frac{x}{2} \right]^{2m+i},$$

that is to say

$$J_{-i}(x) = (-1)^i J_i(x).$$

In a similar way we can derive the well known relations

$$iJ_i(x) = \frac{x}{2} [J_{i-1}(x) + J_{i+1}(x)],$$

$$\frac{dJ_i(x)}{dx} = \frac{1}{2} [J_{i-1}(x) - J_{i+1}(x)],$$

from the following properties of Cauchy's numbers :

$$N_{-p,j+1,q} = N_{-p+1,j,q} + N_{-p-1,j,q}$$

$$N_{-p,j,q+1} = N_{-p+1,j,q} - N_{-p-1,j,q}.$$

As another application of our formulas let us express the function  $\left[ \frac{r}{a} - 1 \right]^m$  of Celestial Mechanics by means of Bessel's functions given the formulas\*

$$\left[ \frac{r}{a} - 1 \right]^m = \frac{1}{2} c_0^{(m)} + \sum_{i=1}^{i=\infty} c_i^{(m)} \cos i\zeta$$

$$c_i^{(m)} = (-1)^m \frac{2m}{i} \left[ \frac{e}{2} \right]^m \sum_{k=0}^{k=\infty} \frac{N_{-i,m-1,i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} \quad (7)$$

\* See *Tisserand, Mécanique Céleste*, tome I, pp. 237-239. For the values of  $c_0^{(m)}$  see l. c. formulas (20) and (21). These values are also given below, namely, by formulas (9) and (10).

To this end we remark first that the number  $N_{-p,j,q}$  can be expressed linearly in Cauchy's numbers of the form  $N_{-p',0,q}$  by the formula\*

$$N_{-p,j,q} = N_{-p+j,0,q} + \left[ \frac{j}{1} \right] N_{-p+j-2,0,q} + \left[ \frac{j}{2} \right] N_{-p+j-4,0,q} + \dots + N_{-p-j,0,q},$$

which becomes in the present case

$$\begin{aligned} N_{-i,m-1,i-m+1+2k} &= N_{-i+m-1,0,i-m+1+2k} \\ &+ \left[ \frac{m-1}{1} \right] N_{-i+m-3,0,i-m+1+2k} \\ &+ \left[ \frac{m-1}{2} \right] N_{-i+m-5,0,i-m+1+2k} \\ &+ \dots \\ &+ N_{-i-m+1,0,i-m+1+2k}. \end{aligned}$$

It must be noticed that in the first line of this formula  $k \geq 0$ , in the second line  $k \geq 1$ , in the third  $k \geq 2$ , and so on; in the last line  $k \geq (m-1)$  because  $N_{-p,j,q} = 0$  when  $-p+j+q$  is negative. Hence

$$\begin{aligned} \sum_{k=0}^{k=\infty} \frac{N_{-i,m-1,i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} &= \sum_{k=0}^{k=\infty} \frac{N_{-i+m-1,0,i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} \\ &+ \left[ \frac{m-1}{1} \right] \sum_{k=1}^{k=\infty} \frac{N_{-i+m-3,0,i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} \\ &+ \left[ \frac{m-1}{2} \right] \sum_{k=2}^{k=\infty} \frac{N_{-i+m-5,0,i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} \\ &+ \dots \\ &+ \sum_{k=m-1}^{k=\infty} \frac{N_{-i-m+1,0,i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} \\ &= \sum_{k=0}^{k=\infty} \frac{N_{-i+m-1,0,i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} \\ &+ \left[ \frac{m-1}{1} \right] \sum_{k=0}^{k=\infty} \frac{N_{-i+m-3,0,i-m+3+2k}}{(i-m+2k+2)!} \left[ \frac{ie}{2} \right]^{i-m+2k+2} \end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{m-1}{2} \right] \sum_{k=0}^{k=\infty} \frac{N_{-i+m-5, 0, i-m+5+2k}}{(i-m+2k+4)!} \left[ \frac{ie}{2} \right]^{i-m+2k+4} \\
& + \dots \\
& + \sum_{k=0}^{k=\infty} \frac{N_{-i-m+1, 0, i+m+2k-1}}{(i+m+2k-2)!} \left[ \frac{ie}{2} \right]^{i+m+2k-2} \\
& = \frac{2}{i} \frac{d}{de} \\
& \left\{ J_{i-m+1}(ie) + \left[ \frac{m-1}{1} \right] J_{i-m+3}(ie) + \left[ \frac{m-1}{2} \right] J_{i-m+5}(ie) + \dots + J_{i+m-1}(ie) \right\}
\end{aligned}$$

and therefore

$$\begin{aligned}
c_i^{(m)} &= (-1)^m \frac{4m}{i^2} \left[ \frac{e}{2} \right]^m \frac{d}{de} \\
& \left\{ J_{i-m+1}(ie) + \left[ \frac{m-1}{1} \right] J_{i-m+3}(ie) + \left[ \frac{m-1}{2} \right] J_{i-m+5}(ie) + \dots + J_{i+m-1}(ie) \right\}
\end{aligned}$$

which is the formula we intended to derive.

Conversely, suppose the function  $\left[ \frac{r}{a} - 1 \right]^m$  expressed with the help of Bessel's functions. To obtain this expression directly, we remark that

$$\begin{aligned}
\left[ \frac{r}{a} - 1 \right]^m &= (-1)^m \left[ \frac{e}{2} \right]^m 2 \cdot (2^{m-1} \cos mu) \\
&= (-1)^m \left[ \frac{e}{2} \right]^m \\
& 2 \left\{ \cos mu + \left[ \frac{m}{1} \right] \cos(m-2)u + \left[ \frac{m}{2} \right] \cos(m-4)u + \dots \right\}
\end{aligned}$$

the last term within the brackets being  $\frac{1}{2} \left[ \frac{2m_1}{m_1} \right]$  if  $m = 2m_1$ ; and  $\left[ \frac{2m_1+1}{m_1} \right]$   $\cos u$  if  $m = 2m_1 + 1$ . In either case the above development will contain a constant term which we will denote by  $\frac{1}{2} c_0^{(m)}$ . When  $m = 2m_1$

$$\frac{1}{2} c_0^{(m)} = (-1)^m \left[ \frac{e}{2} \right]^m \left[ \frac{2m_1}{m_1} \right] = \frac{(m_1+1)(m_1+2) \dots 2m_1}{m_1!} \left[ \frac{e}{2} \right]^{2m_1} \quad (9)$$

when  $m = 2m_1 + 1$ , since  $\cos u = -\frac{1}{2} e + \text{periodic terms}$ , we shall have the constant term

$$\frac{1}{2} c_0^{(m)} = -\frac{1}{2} e (-1)^m \left[ \frac{e}{2} \right]^m 2 \left[ \frac{2m_1+1}{m_1} \right] = 2 \frac{(m_1+2)(m_1+3) \dots (2m_1+1)}{m_1!} \left[ \frac{e}{2} \right]^{m+1}. \quad (10)$$

Formulas (9) and (10) are identical with (20) and (21) of Tisserand (l. c. p. 238).

We have on the other hand for  $m > 1$

$$\cos mu = m \sum_{i=1}^{i=\infty} [J_{i-m}(ie) - J_{i+m}(ie)] \frac{\cos i\zeta}{i},$$

which formula holds also for  $m = 1$  if we leave out the non-periodic term. Hence

$$\left[ \frac{r}{a} - 1 \right]^m = \frac{1}{2} c_0^{(m)} + (-1)^m \left[ \frac{e}{2} \right]^m 2 \sum_{i=1}^{i=\infty} A_i \frac{\cos i\zeta}{i}, \quad (11)$$

where we have put

$$\begin{aligned} A_i = m \left\{ J_{i-m}(ie) - J_{i+m}(ie) \right\} + (m-2) \left[ \frac{m}{1} \right] \left\{ J_{i-m+2}(ie) - J_{i+m-2}(ie) \right\} \\ + (m-4) \left[ \frac{m}{2} \right] \left\{ J_{i-m+4}(ie) - J_{i+m-4}(ie) \right\} + \dots \end{aligned}$$

Now, it can be readily shown that

$$(m-2p) \left[ \frac{m}{p} \right] = m \left\{ \left[ \frac{m-1}{p} \right] - \left[ \frac{m-1}{p-1} \right] \right\}$$

so that

$$\begin{aligned} A_i = m \left\{ [J_{i-m}(ie) - J_{i-m+2}(ie)] + \left[ \frac{m-1}{1} \right] [J_{i-m+2}(ie) - J_{i-m+4}(ie)] + \dots \right\} \\ = \frac{2m}{i} \frac{d}{de} \left\{ J_{i-m+1}(ie) + \left[ \frac{m-1}{1} \right] J_{i-m+3}(ie) + \dots \right\}, \end{aligned}$$

and, therefore,

$$\begin{aligned} \left[ \frac{r}{a} - 1 \right]^m = \frac{1}{2} c_0^{(m)} + (-1)^m 4m \left[ \frac{e}{2} \right]^m \sum_{i=1}^{i=\infty} \frac{\cos i\zeta}{i^2} \frac{d}{de} \\ \left\{ J_{i-m+1}(ie) + \left[ \frac{m-1}{1} \right] J_{i-m+3}(ie) + \dots \right\} \end{aligned}$$

We have shown above that

$$\begin{aligned} \frac{d}{de} \left\{ J_{i-m+1}(ie) + \left[ \frac{m-1}{1} \right] J_{i-m+3}(ie) + \dots \right\} \\ = \frac{i}{2} \sum_{k=0}^{k=\infty} \frac{N_{-i, m-1, i-m+1+2k}}{(i-m+2k)!} \left[ \frac{ie}{2} \right]^{i-m+2k} \end{aligned}$$

Hence,  $c_i^{(m)}$  being given by formula (7) we shall have

$$\left[ \frac{r}{a} - 1 \right]^m = \frac{1}{2} c_0^{(m)} + \sum_{i=1}^{i=\infty} c_i^{(m)} \cos i\zeta$$

which solves the proposed problem.